Supplemental Materials for Gregory E. Goering and John R. Boyce, "Emissions Taxation in Durable Goods Oligopoly," *The Journal of Industrial Economics* 47 (1), March 1999, pp. 125-143

Optimal Short-Run Durability

The main area of interest in durability models in the short-run is the conditions under which the firm's durability path is declining over time. For example, White [1971] argues that US automobile manufactures tended to decrease the durability of their products since the 1940's. Using a monopoly model Muller and Peles [1990] show that this can occur for specific forms of the manufacturer's cost function. Equations (7) and (8) indicate that in oligopolistic industries the durability time path also depends critically upon taxation. As the tax/emissions structure is changed it alters each firm's total cost function (manufacturing *plus* tax costs). Consequently, the tax/emission structure influences the firm's marginal cost and the evolution of durability over time. If, for example, an excise tax is placed on the industry [i.e. $\varepsilon(\delta, q) = q$], then (8) in the text simplifies to

$$(8') \qquad q\int_t^{\infty}\phi_\delta[s-t,\delta(t)]\{f[nQ(s)] + f'[nQ(s)]Q(s)\}e^{-r(s-t)}ds \; - c'(\delta)q \;\; = 0.$$

To make a definitive statement about the time path of durability we need to know the behavior of the stock. It is standard (e.g., Muller and Peles [1990]) to assume that the firm's stock increases over time, so $\dot{Q} > 0$.[1] Then the following can be stated (e.g., Muller and Peles [1990]):

PROPOSITION: *If the stock of each firm increases over time* ($\dot{Q} > 0$)*, then durability will decrease over time* ($\dot{\delta} < 0$) *with an excise tax.*

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[1] A dot over the variables denotes the derivative with respect to time, e.g., $\dot{Q} = dQ/dt$.

Proof: For notational ease let $R(t) = f[nQ(t)]+f'[nQ(t)]Q(t)$. Implicitly differentiating (8') with respect to time then gives:

$$\dot{\delta} = \frac{\zeta}{\alpha} \ ,$$

where,

$$\zeta = \phi_\delta[0,\delta(t)]R(t) - \int_t^\infty \{r\phi_\delta[s-t,\delta(t)] - \phi_{\delta,t-s}[s-t,\delta(t)]\}R(s)e^{-r(s-t)}ds,$$

and,

$$\alpha \ = \ \int_t^\infty R(s)\phi_{\delta\delta}[s-t,\delta(t)]e^{-r(s-t)}ds - c''(\delta(t)).$$

If the equilibria are stable then α will have the same sign as the second-direct partial of the Hamiltonian with respect to durability $H_{\delta\delta}$, implying that $\alpha < 0$ (see the discussion in section 3 footnote 14 concerning the relationship between second-order conditions and stability). Thus the sign of α is determined by the sign of ζ.

Note that in most cases $\phi_\delta[0, \delta(t)] = 0$ since durability does not impact the survival or decay process at the current instant even at the margin. For example, if we suppose that the product is subject to exponential decay it indicates $\phi[t\text{–s},\delta_i(s)] = \exp[-(t-s)/\delta_i(s)]$. Thus $\phi_\delta[0, \delta(t)] = 0$, implying the first term in ζ vanishes. Muller and Peles (1990) use this functional form to further demonstrate that ζ is positive as long as the stock of output produced by a monopolist is increasing overtime ($\dot{Q} > 0$). Their proof holds in our oligopoly model as well, implying that if the stock of industry output increases as t increases then $\zeta > 0$. Thus exponential product decay and $\varepsilon_\delta = 0$ (emissions are due to output rather than durability) are sufficient to ensure $\dot{\delta} < 0$ so the oligopolist's durability will decrease over time. •

The intuition behind this result is that the oligopolist's durability decreases over time ($\dot{\delta} < 0$) since the revenue attributable to a unit of durability decreases over time. In other words, the value of a unit of durability decreases over time with no corresponding shift in the firm's manufacturing and tax costs at the margin, so with this type of tax structure durability tends to decline as White (1971) found for US automobile manufacturers. This decline in durability would still occur if emissions taxes were levied as long as the emissions of the firm are due to quantity rather than durability (i.e., $\varepsilon_\delta = 0$). Thus, the existence of an excise tax, or an emissions tax with certain specific forms of the emissions function, indicates $\dot{\delta} < 0$ is still possible. This can occur regardless of the number of firms in the industry, suggesting the observed pattern of quality decline found in some durable goods industries (such as the US automobile industry from the nineteen forties through the nineteen sixties) may be, at least in part, related to the type of tax structure in existence.

Appendix A: Derivation of Equations (11) and (12).

In the steady-state, profits to firm i given n firms are in the industry and that the tax rate on emissions equals w are:

$$\text{(A.1)} \qquad \pi^i(\bar{\delta}_i,\bar{q}_i;n,w) \;=\; f\Big(\sum_{j=1}^{n} \bar{Q}_j\Big)\,\bar{Q}_i \;-\; c(\bar{\delta}_i)\,\bar{q}_i \;-\; \varepsilon(\bar{\delta}_i,\bar{q}_i)w,$$

where,

$$\text{(A.2)} \qquad \bar{Q}_i \equiv \rho(\bar{\delta}_i)\,\bar{q}_i, \qquad\qquad i = 1,\ldots,n,$$

is the steady state stock level of the durable good for firm i for derivative purposes (see Sieper and Swan (1973) and footnote 15 in the text). The profit maximizing levels of $\bar{\delta}_i$ and $\bar{q}_i$ (and $\bar{Q}$) are given as the solution to the set of first-order conditions (9) and (10). These equations can be rewritten as the first-order conditions to maximizing (A.1), recognizing that the steady-state stock $\bar{Q}$ depends upon both production $\bar{q}_i$ and durability $\bar{\delta}_i$ through (A.2). Thus, (9) and (10) may be rewritten as (letting π_x denote the partial derivative of π with respect to x, for $x = \bar{q}_i$, $\bar{\delta}_i$, n, and w):

$$\text{(A.3)} \qquad \pi^i_{q_i} \;=\; \Big[f\Big(\sum_{j=1}^{n} \bar{Q}_j\Big) + f'\Big(\sum_{j=1}^{n} \bar{Q}_j\Big)\bar{Q}_i\Big](\partial\bar{Q}_i/\partial\bar{q}_i) \;-\; c(\bar{\delta}_i) \;-\; \varepsilon_{q_i}(\bar{\delta}_i,\bar{q}_i)w \;=\; 0$$

$$\text{(A.4)} \qquad \pi^i_{\delta_i} \;=\; \Big[f\Big(\sum_{j=1}^{n} \bar{Q}_j\Big) + f'\Big(\sum_{j=1}^{n} \bar{Q}_j\Big)\bar{Q}_i\Big](\partial\bar{Q}_i/\partial\bar{\delta}_i) \;-\; c'(\bar{\delta}_i)\bar{q}_i \;-\; \varepsilon_{\delta_i}(\bar{\delta}_i,\bar{q}_i)w \;=\; 0.$$

Given (A.2), $\partial\bar{Q}_i/\partial\bar{\delta}_i = \rho'(\bar{\delta})\bar{q}$ and $\partial\bar{Q}_i/\partial\bar{q}_i = \rho(\bar{\delta}_i)$, and given the assumption of identical firms and $\sum_{j=1}^{n} \bar{Q}_j = n\bar{Q}$ with a symmetric equilibrium, these equations may be rewritten as:

(A.3') $$\pi_q = [f(n\bar{Q}) + f'(n\bar{Q})\bar{Q}]\rho(\bar{\delta}) - c(\bar{\delta}) - \varepsilon_q(\bar{\delta},\bar{q})w = 0,$$

(A.4') $$\pi_\delta = [f(n\bar{Q}) + f'(n\bar{Q})\bar{Q}]\rho'(\bar{\delta})\bar{q} - c'(\bar{\delta})\bar{q} - \varepsilon_\delta(\bar{\delta},\bar{q})w = 0,$$

If the firms are identical, then the first- and second-order conditions are symmetric. Thus the elements of the Jacobian matrix may be rewritten as:[28]

(A.5) $$\pi_{qq} = [(n+1)f'(n\bar{Q}) + \bar{Q}nf''(n\bar{Q})][\rho(\bar{\delta})]^2 - \varepsilon_{qq}(\bar{\delta},\bar{q})w,$$

(A.6) $$\pi_{q\delta} = [(n+1)f'(n\bar{Q}) + \bar{Q}nf''(n\bar{Q})]\rho(\bar{\delta})\rho'(\bar{\delta})\bar{q} + [f(n\bar{Q}) + \bar{Q}f'(n\bar{Q})]\rho'(\bar{\delta}) \,,$$
$$- c'(\bar{\delta}) - \varepsilon_{qd}(\bar{\delta},\bar{q})w$$

(A.7) $$\pi_{\delta\delta} = [(n+1)f'(n\bar{Q}) + \bar{Q}nf''(n\bar{Q})][\rho'(\bar{\delta})\bar{q}]^2 + [f(n\bar{Q}) + \bar{Q}f'(n\bar{Q})]\rho''(\bar{\delta})\bar{q} \,,$$
$$- c''(\bar{\delta})\bar{q} - \varepsilon_{dd}(\bar{\delta},\bar{q})w.$$

Furthermore, the derivatives with respect to the arguments *n* and *w* are:

(A.8) $$\pi_{qw} = -\varepsilon_q(\bar{\delta},\bar{q}),$$

(A.9) $$\pi_{qn} = [f'(n\bar{Q}) + f''(n\bar{Q})\bar{Q}]\bar{Q}\rho(\bar{\delta}),$$

(A.10) $$\pi_{\delta w} = -\varepsilon_d(\bar{\delta},\bar{q}),$$

(A.11) $$\pi_{\delta n} = [f'(n\bar{Q}) + f''(n\bar{Q})\bar{Q}]\bar{Q}\rho'(\bar{\delta})\bar{q}.$$

As the maximizing levels of $\bar{\delta}_i$ and $\bar{q}_i$ are given as the solution to (A.3') and (A.4), (A.5) through (A.11) can be used to determine the effects of changes in either *w*, the emissions tax, or *n*, the number of firms in the industry on these equilibrium choices.

In matrix notation the total differentials of the system (A.3') and (A.4') may be written as:

$$\text{(A.12)} \qquad \begin{pmatrix} \pi_{qq} & \pi_{q\delta} \\ \pi_{\delta q} & \pi_{\mathrm{dd}} \end{pmatrix} \begin{pmatrix} d\bar{q} \\ d\bar{\delta} \end{pmatrix} = -\begin{pmatrix} \pi_{qw} \\ \pi_{\delta w} \end{pmatrix} dw - \begin{pmatrix} \pi_{qn} \\ \pi_{\delta n} \end{pmatrix} dn.$$

where π_{ij} $i = \bar{q}, \bar{\delta}$, $j = \bar{q}, \bar{\delta}, n, w$, are defined by (A.5) through (A.11). Let the Jacobian matrix be denoted as $\boldsymbol{J}$. It is assumed that the determinant of $\boldsymbol{J}$ is positive.

The remainder of the analysis involves using Cramer's rule to obtain the necessary derivatives. The derivatives of $\bar{\delta}$ with respect to *n* and *w* are:[29]

$$\text{(A.13)} \qquad \frac{\partial \bar{\delta}}{\partial x} = \mathrm{Det}\begin{pmatrix} \pi_{qq} & -\pi_{qx} \\ \pi_{\mathrm{d}q} & -\pi_{\mathrm{d}x} \end{pmatrix} / |\boldsymbol{J}| = (\pi_{qx}\pi_{\delta q} - \pi_{qq}\pi_{\delta x})/|\boldsymbol{J}|, \quad x = n, w,$$

where Det[$\boldsymbol{A}$] and |$\boldsymbol{A}$| both mean the determinant of $\boldsymbol{A}$ for any square matrix $\boldsymbol{A}$.

Consider the derivation of $\frac{\partial \bar{\delta}}{\partial n}$ (equation 11 in the text). From (A.13), and dropping the arguments of each function,

$$\frac{\partial \bar{\delta}}{\partial n} = (\pi_{qn}\pi_{\delta q} - \pi_{qq}\pi_{\delta n})/|\boldsymbol{J}|$$

$$= \bar{Q}[f' + \bar{Q}f'']\{[(n+1)f' + n\bar{Q}f''](\rho^2\rho'\bar{q} - \rho^2\rho'\bar{q}) + (\bar{Q}f' + f)\rho'\rho$$

$$- \rho c' - \rho\varepsilon_{\delta q}w + \rho'\bar{q}\varepsilon_{qq}w\}/|\boldsymbol{J}|$$

$$= \bar{Q}[f' + \bar{Q}f''][(c' + e_\delta w/\bar{q})\rho - \rho c' - \rho\varepsilon_{\delta q}w + \bar{q}\rho'\varepsilon_{qq}w]/|\boldsymbol{J}|$$

$$\text{(A.14)} \qquad = w\bar{Q}[f' + \bar{Q}f''][\bar{q}\rho'\varepsilon_{qq} + \rho(\varepsilon_\delta/\bar{q} - \varepsilon_{\delta q})]/|\boldsymbol{J}|,$$

where the second equality substitutes in the π_{ij} terms, the third equality cancels redundant terms and uses $\pi_\delta = 0$ from (A.4) to eliminate the $(\bar{Q}f' + f)\rho'$ term. It can be seen that (A.14) equals equation (11) from the text.

Next, consider the derivation of the change in the optimal durability level with respect to a change in the emissions tax w (equation (12) in the text). From (A.13),

$$\frac{\partial\bar{\delta}}{\partial w} = (\pi_{qw}\pi_{\delta q} - \pi_{qq}\pi_{\delta w})/|\boldsymbol{J}|$$

$$= (\varepsilon_\delta\{[(n+1)f' + n\bar{Q}f'']\rho^2 - \varepsilon_{qq}w\} - \varepsilon_q\{[(n+1)f' + \bar{Q}f'']\rho\rho'\bar{q}$$

$$+ (\bar{Q}f' + f)\rho' - c' - \varepsilon_{\delta q}w\})/|\boldsymbol{J}|$$

$$= \{\rho[(n+1)f' + n\bar{Q}f''](\varepsilon_\delta\rho - \varepsilon_q\rho'\bar{q}) - \varepsilon_\delta\varepsilon_{qq}w - (\bar{Q}f' + f)\rho'\varepsilon_q$$

$$+ \varepsilon_q c' + \varepsilon_q\varepsilon_{\delta q}w]\}/|\boldsymbol{J}|$$

$$= \{\rho[(n+1)f' + n\bar{Q}f''](\varepsilon_\delta\rho - \varepsilon_q\rho'\bar{q}) - \varepsilon_\delta\varepsilon_{qq}w - (c' + \varepsilon_\delta w/\bar{q})\varepsilon_q$$

$$+ \varepsilon_q c' + \varepsilon_q\varepsilon_{\delta q}w\}/|\boldsymbol{J}|$$

$$\text{(A.15)} \qquad = \{\rho[(n+1)f' + n\bar{Q}f''](\varepsilon_\delta\rho - \varepsilon_q\rho'\bar{q}) - [(\varepsilon_q\varepsilon_\delta)/\bar{q} + \varepsilon_\delta\varepsilon_{qq} - \varepsilon_q\varepsilon_{\delta q}]w\}/|\boldsymbol{J}|,$$

where the second equality substitutes in for the π_{ij} terms, the third equality collects terms, the fourth uses $\pi_\delta = 0$ to substitute out the $(\bar{Q}f' + f)$ term, and the final equality cancels the $c'\varepsilon_q$ terms. Equation (12) from the text equals (A.15).

Appendix B: Derivation of Equations (16) and (17).

The social planner's problem involves maximizing (15), which may be re-written as:

$$\text{(B.1)}\quad V(w) = \int_0^{\sum_{j=1}^{n} \bar{Q}_j(w)} f(g)dg - \sum_{j=1}^{n} c[\bar{\delta}_j(w)]\bar{q}_j(w) - \sum_{j=1}^{n} \varepsilon[\bar{\delta}_j(w),\bar{q}_j(w)].$$

The value of w which maximizes (B.1) must satisfy (dropping the arguments from the functions):

$$\text{(B.2)}\quad V'(w) = f(\sum_{j=1}^{n} \bar{Q}_j) \sum_{j=1}^{n} \left(\frac{\partial \bar{Q}_j \partial \bar{q}_j}{\partial q_j\, \partial w} + \frac{\partial \bar{Q}_j \partial \bar{\delta}_j}{\partial \delta_j\, \partial w} \right) - \sum_{j=1}^{n} \left(c(\bar{\delta}_j)\frac{\partial \bar{q}_j}{\partial w} - c'(\bar{\delta}_j)\bar{q}_j \frac{\partial \bar{\delta}_j}{\partial w} \right)$$

$$- \sum_{j=1}^{n} \left(\varepsilon_{\delta_j} \frac{\partial \bar{\delta}_j}{\partial w} + \varepsilon_{q_j} \frac{\partial \bar{q}_j}{\partial w} \right) = 0.$$

When firms are identical and symmetry is imposed (B.2) may be rewritten as,

$$V'(w) = f(n\bar{Q}) \left(\rho(\bar{\delta})\frac{\partial \bar{q}}{\partial w} + \rho'(\bar{\delta})\bar{q}\frac{\partial \bar{\delta}}{\partial w} \right) - \left(c(\bar{\delta})\frac{\partial \bar{q}}{\partial w} + c'(\bar{\delta})\bar{q}\frac{\partial \bar{\delta}}{\partial w} \right)$$

$$- \left(\varepsilon_q \frac{\partial \bar{q}}{\partial w} + \varepsilon_\delta \frac{\partial \bar{\delta}}{\partial w} \right)$$

$$\text{(B.3)}\quad = \left(f(n\bar{Q})\rho(\bar{\delta}) - c(\bar{\delta}) - \varepsilon_q \right)\frac{\partial \bar{q}}{\partial w} + \left(f(n\bar{Q})\rho'(\bar{\delta})\bar{q} - c'(\bar{\delta}) - \varepsilon_\delta \right)\frac{\partial \bar{\delta}}{\partial w} = 0,$$

which verifies equation (16) in the text. To obtain equation (17), note that individual firms choose $\bar{q}$ and $\bar{\delta}$ to maximize (A.1), which means that $\bar{q}$ and $\bar{\delta}$ must jointly solve (A.3') and (A.4') (equations (9) and (10) in the text). Since (A.3') and (A.4') are identities, the equalities hold for all values of *w*, including, of course, the value of *w* which maximizes (B.1). Thus one may subtract $\pi_\delta \equiv 0$ from the term involving $\frac{\partial\bar{\delta}}{\partial w}$ in (B.3) and $\pi_q \equiv 0$ from the term involving $\frac{\partial\bar{q}}{\partial w}$. Thus,

$$
\begin{aligned}
& f(n\bar{Q})\rho(\bar{\delta}) - c(\bar{\delta}) - \varepsilon_q \\
&\quad = f(n\bar{Q})\rho(\bar{\delta}) - c(\bar{\delta}) - \varepsilon_q - \{[f'(n\bar{Q})\bar{Q} + f(n\bar{Q})]\rho(\bar{\delta}) - c(\bar{\delta}) - \varepsilon_q w\} \\
&\quad = \varepsilon_q(w-1) - f'(n\bar{Q})\rho(\bar{\delta})\bar{Q}, \qquad \text{(B.4)}
\end{aligned}
$$

and,

$$
\begin{aligned}
f(n\bar{Q})\rho'(\bar{\delta})\bar{q} - c'(\bar{\delta})\bar{q} - \varepsilon_\delta &= f(n\bar{Q})\rho'(\bar{\delta})\bar{q} - c'(\bar{\delta})\bar{q} - \varepsilon_\delta \\
&\quad - \{[f'(n\bar{Q})\bar{Q} + f(n\bar{Q})]\rho'(\bar{\delta})\bar{q} - c'(\bar{\delta})\bar{q} - \varepsilon_\delta w\} \\
&= \varepsilon_\delta(w-1) - f'(n\bar{Q})\rho(\bar{\delta})\bar{q}\bar{Q}. \qquad \text{(B.5)}
\end{aligned}
$$

Substituting (B.4) and (B.5) into (B.3) yields,

$$
[\varepsilon_q(w-1) - f'(n\bar{Q})\rho(\bar{\delta})\bar{Q}]\frac{\partial\bar{q}}{\partial w} + [\varepsilon_\delta(w-1) - f'(n\bar{Q})\rho'(\bar{\delta})\bar{q}\bar{Q}]\frac{\partial\bar{\delta}}{\partial w} = 0, \qquad \text{(B.6)}
$$

Solving (B.6) for *w* yields,

$$
w^* = \frac{[\varepsilon_q + f'(\bar{Q})\rho(\bar{\delta})\bar{Q}]\frac{\partial\bar{q}}{\partial w} + [\varepsilon_\delta + f'(n\bar{Q})\rho'(\bar{\delta})\bar{q}\bar{Q}]\frac{\partial\bar{\delta}}{\partial w}}{\varepsilon_q\frac{\partial\bar{q}}{\partial w} + \varepsilon_\delta\frac{\partial\bar{\delta}}{\partial w}}, \qquad \text{(B.7)}
$$

which equals (17) in the text.