Supplemental Materials for Natalia Fabra

"TACIT COLLUSION IN REPEATED AUCTIONS:

UNIFORM VERSUS DISCRIMINATORY",

The Journal of Industrial Economics Volume (issue), Month, Year, pp. xxx-yyy

In this note, we provide the formal analysis of the extensions discussed in Section V of the paper: firstly, we allow firms to submit multiple rather than a single bid; and secondly, we allow demand to move over time. In these two cases, we show that the paper's main result, namely, that *uniform-price auctions facilitate collusion more than discriminatory auctions,* is robust.

I. MULTIPLE BIDS

We allow firm i to choose any number of steps in its bidding function up to a maximum of $N < \infty$. Bid steps, measured as a quantity of capacity, can be of any finite size. Let $b_i = (b_{i1}, b_{i2}, ..., b_{iN_i})$, $i = 1, 2$, be a set of price bids for i, with $N_i \leq N$, and indexed such that $b_{in} < b_{in+1}$. The number of capacity units, or steps, in i's bid function is given by the number of different price bids, and these are denoted $k_i = (k_{i1}, k_{i2}, ..., k_{iN_i})$, with $\sum_{n=1}^{N_i} k_{in} = k$. Last, let $(b,k) = [(b_1, k_1), (b_2, k_2)]$ be the profile of price and capacity bids.

Lemma S1. Consider the set of price and quantity bid profiles that yield joint profits $PD(P) \in (2\underline{\pi}, P^m D(P^m)]$. The bid profile(s) that minimizes the maximum between firm 1 and 2's one-shot deviation gains must satisfy:

(i) In the discriminatory auction: profits must be evenly divided among firms, and all despatched units must be bid in at P.

(ii) In the uniform-price auction: if $D(P) \leq k$, profits must be evenly divided among firms, and all despatched units must be bid in at P; otherwise, one firm has to bid all its units at prices no greater than $\frac{\underline{\pi}}{k}$, and the highest accepted bid of the remaining firm must be equal to P.

Proof of Lemma S1:

Discriminatory auction: Consider all the price and capacity bid profiles such that joint profits equal $PD(P)$. If firms submit symmetric bid schedules (i.e. $b_{1n} = b_{2n}$

and $k_{1n} = k_{2n}$ $\forall n \in [1, N_i]$, $i = 1, 2$), then each firm's equilibrium profits are equal to $\frac{1}{2}PD(P)$. Otherwise, there will one firm whose profits exceed $\frac{1}{2}PD(P)$. Let us index such a high-profit firm as firm 1, and let firm i's profits be denoted by $\pi_i(b, k)$, $i = 1, 2$. That is, $\pi_1(b, k) \geq \pi_2(b, k)$, where the inequality is strict for the asymmetric bid profiles. A similar argument to that of Lemma 3 shows that firm 2's deviation gains are equal to firm 1's profits (this is achieved by slightly undercutting each of firm 1's units). Hence, firms 2's one-shot deviation gains are given by,

$$\begin{aligned}\Gamma_2(b, k) &= \pi_1(b, k) - \pi_2(b, k) \\ &= \pi_1(b, k) - [D(P)P - \pi_1(b, k)] \\ &= 2\pi_1(b, k) - D(P)P\end{aligned}$$

Hence, in order to minimize firm 2's one-shot deviation gains, one has to minimize $\pi_1(b, k)$. This is achieved by equating $\pi_1(b, k)$ and $\pi_2(b, k)$, i.e. by letting firms divide joint profits $PD(P)$ equally.

It remains to show that all despatched units must be bid in at the same price. If some despatched units are bid in at prices below the market price, other units must be bid in at prices exceeding P for joint profits to equal $PD(P)$. This would just enhance firms' deviation gains, given that the profits from undercutting the high-priced unit are increasing in its bid.

Uniform-price auction: For the case in which $D(P) \leq k$, the proof follows as in the discriminatory auction. Let $D(P) > k$. If firm i bids all its units below $\frac{\pi}{k}$, the deviation gains of firm j are equal to the minmax, and those of firm i are zero. Clearly, these constitute the lowest levels to which bidders' deviation profits can be driven down to. Last, given that firm i is selling at capacity, the market price will be given by firm j's highest accepted bid. For joint profits to be equal to $D(P)P$, such a bid must equal P.

Corollary S1. The one-shot deviation gains induced by the bid profiles identified in Lemma S1 are equal to those induced by the bid profiles identified in Lemma 3. Hence, Proposition 1 applies independently of whether $N = 1$ or $N > 1$.

Lemma S2. The severity of the optimal penal code is independent of whether $N = 1$ or $N > 1$.

Proof of Lemma S2. If $N = 1$, a firm's profits can be credibly driven down to its minmax under the two auction formats (Proposition 1). Given that a penal code giving a firm per-period profits lower than the minmax is not credible, it follows that setting $N > 1$ would not allow to increase the severity of the optimal penal code.

Lemma S3. The paths of bid profiles identified in Proposition 3 are optimal independently of whether $N = 1$ or $N > 1$. Therefore, Proposition 4 applies.

Proof of Lemma S3. The proof follows exactly as that of Proposition 3.

II. FUTURE DEMAND UNCERTAINTY

Following Rotemberg and Saloner [1986] and Staiger and Wolak [1992], we consider the case in which demand is subject to i.i.d. shocks, θ, which take values in the support $\left[\underline{\theta}, \overline{\theta}\right]$ according to the distribution function $G(\theta)$. We will assume that the parameter θ enters the demand function multiplicatively[1] and that the current demand state is observable at the beginning of each period before firms submit their bids.

A strategy, that is, a contingent plan of action, for firm i in the repeated game *with future demand uncertainty* is an infinite sequence $S_i = (S_i(1), S_i(2), ..., S_i(t), ...)$, where $S_i(1)$ is a function that maps the set of possible demand realizations in period 1 into a determinate initial bid offer, and $S_i(t)$ is a function that maps the market price and the quantities allocated to firm i in periods $1, 2, ..., t-1$ and the set of possible demand realizations at period t into a bid offer $b_i(t)$ for firm i in period t. The strategy profile (S_1, S_2) induces a contingent path of bid profiles $(b(1), b(2), ..., b(t), ...)$, where $b_i(1) = S_i(1)$, and given $\{(b(1), b(2), ..., b(t-1)), (\theta(1), \theta(2), ..., \theta(t))\}$, $b_i(t) = S_i(q_i(b(1); \theta(1)), P(b(1)), ..., q_i(b(t-1); \theta(t-1)), P(b(t-1)), \theta(t))$..

Our aim is to compare the highest level of expected profits that, for a given discount factor, can be sustained under the two auction formats. First, some lemmas are needed.

To understand the effects of future demand uncertainty, let us first suppose that firms decide to collude by submitting identical bids for all possible realizations of

[1]Note that this assumption implies that total industry profits are proportional to θ, and that the monopoly price is independent of θ.

demand. When evaluated at the monopoly price, the incentive compatibility that binds first is the one at which firms' one-shot deviation gains are maximum. Lemma S4 identifies the demand realization at which firms' one-shot deviation gains from the symmetric bid profile giving monopoly profits are the greatest.

Lemma S4. Firms' one-shot deviation gains from the symmetric bid profile $b_i = P^m$, $i = 1, 2$, attain their maximum at $\theta = \widetilde{\theta}$, where $\widetilde{\theta}$ equals $\overline{\theta}$ if $\overline{\theta} < \frac{k}{D(P^m)}$, $\underline{\theta}$ if $\underline{\theta} > \frac{k}{D(P^m)}$ and $\frac{k}{D(P^m)}$ otherwise.

Proof of Lemma S4. For a demand realization θ, a firm's profits at the symmetric bid profile $b_i = P^m$, $i = 1, 2$ are equal to $\frac{1}{2}\theta D(P^m)P^m$. From Lemma 2 we can characterize a firm's profits at its best response to P^m : if $\theta \leq \frac{k}{D(P^m)}$, these are given by $\theta D(P^m)P^m$, and if $\theta > \frac{k}{D(P^m)}$, these are given by kP^m. Hence,

$$\Gamma_i\left(P^m, P^m; \theta\right) = \begin{cases} \frac{1}{2}\theta D(P^m)P^m & \text{if} \quad \theta \leq \frac{k}{D(P^m)} \\ kP^m - \frac{1}{2}\theta D(P^m)P^m & \text{if} \quad \theta > \frac{k}{D(P^m)} \end{cases}$$

and

$$\frac{\partial\Gamma_i(P^m,P^m;\theta)}{\partial\theta} = \begin{cases} \frac{1}{2}D(P^m)P^m & \text{if} \quad \theta \leq \frac{k}{D(P^m)} \\ -\frac{1}{2}D(P^m)P^m & \text{if} \quad \theta > \frac{k}{D(P^m)} \end{cases}$$

Thus, $\Gamma_i\left(P^m, P^m; \theta\right)$ is maximum at $\theta = \widetilde{\theta}$, where $\widetilde{\theta}$ equals $\overline{\theta}$ if $\overline{\theta} < \frac{k}{D(P^m)}$, $\underline{\theta}$ if $\underline{\theta} > \frac{k}{D(P^m)}$ and $\frac{k}{D(P^m)}$ otherwise.

The following lemma provides some existence results:

Lemma S5.

(i) There exists $\underline{\delta} \in \left[0, \frac{1}{2}\right]$ such that there exists a contingent path of symmetric bid profiles that is sustainable for all $\delta \geq \underline{\delta}$.

(ii) There exists $\overline{\delta} \in [\underline{\delta}, 1)$ such that monopoly profits can be sustained through symmetric bidding if and only if $\delta \geq \overline{\delta}$.

Proof of Lemma S5. Define $\underline{\delta}\left(\theta\right)$ as the critical discount factor above which there exists an equilibrium in which firms' bids are symmetric at the demand realization θ, assuming that firms bid symmetrically for all the remaining θ realizations. Hence, $\underline{\delta} = \max_{\theta\in\left[\underline{\theta},\overline{\theta}\right]} \underline{\delta}\left(\theta\right)$. Following the proof of Lemma 4, $\underline{\delta}\left(\theta\right) = 0$ for θ such that $\theta D(0) \leq k$ or such that $P^r\left(\theta\right) = \theta D^{-1}(2k)$; $\underline{\delta}\left(\theta\right) = \frac{1}{2}$ otherwise.

Define $\overline{\delta}(\theta)$ as the critical discount factor above which monopoly pricing can be sustained through symmetric bidding at all demand realizations. Hence, $\overline{\delta} = \max_{\theta\in[\underline{\theta},\overline{\theta}]} \overline{\delta}(\theta)$. Since the *i.i.d.* assumption implies that the losses from cheating are equal at all demand realizations, it follows that $\overline{\delta}$ will be associated with the demand realization at which the one-shot deviation gains from monopoly pricing are the greatest. Following the previous Lemma, $\overline{\delta} = \overline{\delta}\left(\widetilde{\theta}\right)$. Following the main text, $\overline{\delta} = \frac{1}{2}$ if $\widetilde{\theta}D(0) \leq k$, $\overline{\delta} = 0$ if $P^r\left(\widetilde{\theta}\right) = \widetilde{\theta}D^{-1}(2k)$ and $\overline{\delta} \in (\underline{\delta}, 1)$ otherwise.

We can now prove the following Proposition.

Proposition S1. For given $\delta \in (0,1)$, the highest sustainable profit level in the uniform-price auction (weakly) exceeds the one in the discriminatory auction; if peak demand at the monopoly price is at least equal to the capacity of a single firm and if monopoly profits are not sustainable in the discriminatory auction, the comparison is strict.

Proof of Proposition S1. For given $\delta \in (0,1)$, let $P^*(\theta)$ be the highest sustainable price under symmetric bidding for the demand realization $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$. For $\delta \in \left[\overline{\delta}, 1\right)$, $P^*(\theta) = P^m\ \forall\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ (Lemma S5 (ii)); for $\delta \in (0, \underline{\delta})$, $P^*(\theta) < P^r(\theta)$ (Lemma S5 (i)); for $\delta \in \left[\underline{\delta}, \overline{\delta}\right)$, $P^*(\theta)$ is the highest price that satisfies

$$P^*(\theta)\min\{k, \theta D(P^*(\theta))\} - \frac{\theta D(P^*(\theta))P^*(\theta)}{2} = \frac{\delta}{1-\delta}\int_{\underline{\theta}}^{\overline{\theta}}\left[\frac{\theta D(P^*(\theta))P^*(\theta)}{2} - \underline{\pi}(\theta)\right]dG(\theta)$$

The highest sustainable profit level in the uniform-price auction is at least equal to $\theta D(P^*(\theta))P^*(\theta)$ given that firms may choose to collude on the path of symmetric bid profiles.

The highest sustainable profit level in the uniform-price auction is greater than $\theta D(P^*(\theta))P^*(\theta)$ if $k < \overline{\theta}D(P^m)$ and $\delta \in \left(0, \overline{\delta}\right)$: as defined above, $\widetilde{\theta} = \max\left\{\underline{\theta}, \frac{k}{D(P^m)}\right\}$; given that $P^*\left(\widetilde{\theta}\right) < P^m$ for $\delta \in \left(0, \overline{\delta}\right)$, it follows that $\widetilde{\theta}D\left(P^*\left(\widetilde{\theta}\right)\right) > k$. The remainder of the argument follows as in the proof of Proposition 4.

From this we can conclude that:

Corollary S2. In the presence of future demand uncertainty, the comparison between uniform-price and discriminatory auctions in terms of the highest collusive profit level is strengthened.

Proof of Corollary S2. To illustrate this corollary, assume $k < \overline{\theta}D(P^m)$ and $\delta \in \left(0, \overline{\delta}\right)$, so that $P^*\left(\widetilde{\theta}\right) < P^m$. Consider now a demand realization $\widehat{\theta}$ such that $\widehat{\theta}D(0) \leq k$. Without future demand uncertainty, the highest collusive price at $\widehat{\theta} \in \left[\underline{\theta}, \overline{\theta}\right]$ would coincide in the uniform-price and discriminatory auctions (Proposition 4). Let us now consider the case of future demand uncertainty. For a given price $P\left(\widehat{\theta}\right)$, firms' one shot deviation gains at $\widehat{\theta}$ would coincide across auction formats (given that symmetric bidding is optimal in the uniform-price auction for such $\widehat{\theta}$, Lemma 3). However, the losses from cheating at $\widehat{\theta}$ are greater under the uniform-price auction, given that collusive profits at $\widetilde{\theta}$ are greater under the uniform-price auction than under the discriminatory auction (and possibly at other demand realizations as well). Therefore, the highest collusive profit level in the uniform-price auction at the demand realization $\widehat{\theta}$ exceeds $\widehat{\theta}D\left(P^*\left(\widehat{\theta}\right)\right)P^*\left(\widehat{\theta}\right)$. For this same reasoning, the same is true for all demand realizations $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ such that $P^*(\theta) < P^m$.